Disproving Conjecture 247 for all degrees at least eight, with equality at degrees six and seven

Laurence E. Day*

24 July 2026

Abstract

Conjecture 247 of the Graffiti.pc list *Written on the Wall II* (February 2007) asserts that every simple connected regular graph G satisfies $\gamma_t(G) \geq 2p(G)$, where γ_t is the total domination number and p is the minimum number of pairwise vertex-disjoint paths covering the vertex set. The conjecture holds for degrees at most 3 and is recorded as open. We show that it fails for every degree $d \geq 8$, exhibiting two families of counterexamples with exactly computed parameters: a one-hub construction that is $2m$-regular on $2m^2 + m + 1$ vertices with $\gamma_t = m+1$ and $p = m-1$, and a two-hub construction that is d-regular on d^2+1 vertices with $\gamma_t = d+1$ and $p = d-3$. The additive gap $2p - \gamma_t$ is unbounded over each family. The smallest counterexample obtained is 8-regular on 37 vertices, with $\gamma_t = 5 < 6 = 2p$. Both families meet the conjectured bound with equality at their thresholds, giving 6-regular and 7-regular graphs with $\gamma_t = 2p$ and $p \geq 2$; the cases $d \in \{4, 5, 6, 7\}$ remain open.

1 Introduction

A set $D \subseteq V(G)$ is a *total dominating set* of a graph G without isolated vertices if every vertex of G, including every vertex of D itself, has a neighbour in D; the *total domination number* $\gamma_t(G)$ is the minimum size of such a set [1, 5]. A *path cover* of G is a collection of pairwise vertex-disjoint paths in G whose union contains every vertex; single vertices count as paths. The *path cover number* $p(G)$ is the minimum number of paths in a path cover, so $p(G) = 1$ exactly when G is traceable.

Conjecture 247 of the list *Written on the Wall II*, generated by E. DeLaViña's program Graffiti.pc on February 23, 2007, reads as follows [3].

> **Conjecture 247.** If G is a simple connected degree-regular graph, then $\gamma_t(G) \geq 2p(G)$.

Regularity cannot be dropped: the star $K_{1,n}$ has $\gamma_t = 2$ but $p = n-1$, so $\gamma_t < 2p$ once $n \geq 3$, as West's commentary observes [7]. The cases $d \leq 2$ are elementary, and Liu and West proved the conjecture for cubic graphs; a proof appears in [2]. West [7] observes that the cubic case also follows from Reed's theorem [6] that $p(G) \leq (n+2)/9$ for every connected cubic graph G on n vertices, combined with the trivial bound $\gamma_t(G) \geq n/3$. The conjecture is recorded as open in the maintained listing as of 24 July 2026 [3].

In this note we show that the conjecture fails for every degree $d \geq 8$.

*Unaffiliated. Newcastle upon Tyne, United Kingdom. BSc (Jt Hons) Mathematics & Computer Science and PhD Computer Science, both University of Nottingham. Email: l.e.day@outlook.com.

Theorem 1. *For every integer $d \geq 8$ there exists a simple connected d-regular graph G with $\gamma_t(G) < 2p(G)$. Specifically:*

(a) *for even $d = 2m \geq 8$, the one-hub graph $G_{\mathrm{A}}(m)$ of Section 3 has $2m^2 + m + 1$ vertices, $\gamma_t = m + 1$ and $p = m - 1$;*

(b) *for every $d \geq 8$, the two-hub graph $G_{\mathrm{B}}(d)$ of Section 4 has $d^2 + 1$ vertices, $\gamma_t = d + 1$ and $p = d - 3$.*

The smallest counterexample obtained is $G_{\mathrm{A}}(4)$, an 8-regular graph on 37 vertices with $\gamma_t(G_{\mathrm{A}}(4)) = 5 < 6 = 2p(G_{\mathrm{A}}(4))$.

Combining Theorem 1 with the results above, the status of the conjecture by degree is:

degree d	status
$d \leq 3$	true [2]
$d = 4, 5$	open
$d = 6$	open; equality attained with $p = 2$ (Corollary 1)
$d = 7$	open; equality attained with $p = 4$ (Corollary 2)
$d \geq 8$	false (Theorem 1)

Both constructions follow the same principle: total domination is made as efficient as the degree permits, while one small separator forces every path cover to use many paths. All parameter values below are exact, and each has been confirmed by an independent computer verification described in Section 5.

2 Two lemmas

Both lemmas are standard; short proofs are included to keep the note self-contained.

Lemma 1. *If G is an r-regular graph on n vertices with $r \geq 1$, then $\gamma_t(G) \geq \lceil n/r \rceil$.*

Proof. Let D be a total dominating set. Every vertex of G has a neighbour in D, so the open neighbourhoods of the vertices of D cover $V(G)$. Counting incidences, $n \leq \sum_{u \in D} |N(u)| = r|D|$. This bound is recorded in West's commentary [7]; see also [5]. □

Lemma 2. *For every graph G and every $S \subseteq V(G)$,*

$$p(G) \;\geq\; c(G - S) - |S|,$$

where $c(H)$ denotes the number of connected components of H.

Proof. Let $P_1, \ldots, P_q$ be a path cover of G. Delete the vertices of S one at a time; removing a single vertex from a disjoint union of paths increases the number of nonempty path segments by at most one. Hence after deleting all of S, at most $q + |S|$ path segments remain, and they cover every vertex of $G - S$. Each segment lies inside a single component of $G - S$, and each component contains at least one segment, so $c(G - S) \leq q + |S|$. Minimising over path covers gives the claim. □

3 The one-hub family

Fix an integer $m \ge 2$. For $i \in \{1, \dots, m\}$ let

$$B_i = \{a_i,\ b_i,\ c_{i,1}, \dots, c_{i,2m-1}\}$$

be disjoint sets of $2m+1$ vertices, and place on each B_i all edges of the complete graph K_{2m+1} except $a_i b_i$. Add one further vertex x (the *hub*) adjacent to a_i and b_i for every i, and no other edges. Call the resulting graph $G_{\mathrm{A}}(m)$.

Proposition 1. *$G_{\mathrm{A}}(m)$ is connected and $2m$-regular, with $|V(G_{\mathrm{A}}(m))| = 2m^2 + m + 1$.*

Proof. Inside $K_{2m+1} - a_i b_i$ the vertices a_i, b_i have degree $2m-1$, raised to $2m$ by their edge to x; each $c_{i,j}$ keeps all $2m$ neighbours in its block; and x has the $2m$ neighbours $a_1, b_1, \dots, a_m, b_m$. Each block is connected and meets x, so $G_{\mathrm{A}}(m)$ is connected, and $|V(G_{\mathrm{A}}(m))| = 1+m(2m+1) = 2m^2 + m + 1$. $\square$

Proposition 2. $\gamma_t(G_{\mathrm{A}}(m)) = m + 1$.

Proof. The set $D = \{x, a_1, \dots, a_m\}$ is total dominating: x is adjacent to $a_1 \in D$; every a_i and every b_i is adjacent to $x \in D$; and every $c_{i,j}$ is adjacent to $a_i \in D$. Hence $\gamma_t(G_{\mathrm{A}}(m)) \le m+1$. Conversely, by Proposition 1 and Lemma 1,

$$\gamma_t(G_{\mathrm{A}}(m)) \ \ge\ \left\lceil \frac{2m^2+m+1}{2m} \right\rceil = \left\lceil m + \tfrac{1}{2} + \tfrac{1}{2m} \right\rceil = m+1. \qquad \square$$

Proposition 3. $p(G_{\mathrm{A}}(m)) = m - 1$.

Proof. Deleting x leaves exactly the m components $B_1, \dots, B_m$, so Lemma 2 with $S = \{x\}$ gives $p(G_{\mathrm{A}}(m)) \ge m-1$. For the upper bound, take the path

$$P_1 : \quad b_1,\ c_{1,1}, \dots, c_{1,2m-1},\ a_1,\ x,\ a_2,\ c_{2,1}, \dots, c_{2,2m-1},\ b_2,$$

together with, for each $i \in \{3, \dots, m\}$, the path

$$P_{i-1} : \quad b_i,\ c_{i,1}, \dots, c_{i,2m-1},\ a_i.$$

Within each block the only missing edge is $a_i b_i$, and no listed path uses that pair consecutively, so every consecutive pair above is adjacent. These $m-1$ paths are pairwise vertex-disjoint and cover $V(G_{\mathrm{A}}(m))$. $\square$

Corollary 1. *For $m \ge 4$ the graph $G_{\mathrm{A}}(m)$ is a connected $2m$-regular graph with*

$$\gamma_t(G_{\mathrm{A}}(m)) = m+1 \ <\ 2m-2 \ =\ 2p(G_{\mathrm{A}}(m)),$$

so Conjecture 247 fails for every even degree $d = 2m \ge 8$. At the boundary, $G_{\mathrm{A}}(3)$ is 6-regular on 22 vertices with $\gamma_t = 4 = 2p$ and $p = 2$, and $G_{\mathrm{A}}(2)$ is 4-regular on 11 vertices with $\gamma_t = 3 > 2 = 2p$.

Proof. The inequality $m+1 < 2(m-1)$ is equivalent to $m \ge 4$; the boundary values follow from Propositions 1–3 with $m = 3$ and $m = 2$. $\square$

In particular $G_{\mathrm{A}}(4)$ is 8-regular on 37 vertices with

$$\boxed{\gamma_t(G_{\mathrm{A}}(4)) = 5 \ <\ 6 = 2p(G_{\mathrm{A}}(4)).}$$

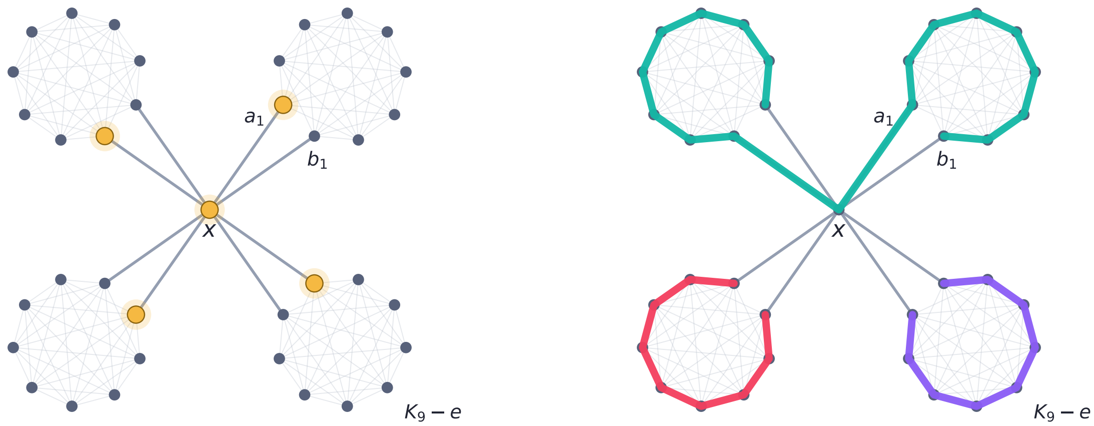

Figure 1: The 37-vertex graph $G_{\mathrm{A}}(4)$ (four blocks $K_9 - a_ib_i$ and the hub x). Left: the total dominating set $\{x, a_1, a_2, a_3, a_4\}$ of Proposition 2. Right: the optimal path cover by three paths of Proposition 3; since $G_{\mathrm{A}}(4) - x$ has four components, two paths cannot suffice (Lemma 2).

4 The two-hub family

The one-hub construction requires an even degree, since the hub meets each block twice. A variant with two hubs removes this restriction and covers all degrees $d \geq 8$, including odd ones.

Fix an integer $d \geq 4$. For $i \in \{1, \ldots, d-1\}$ let

$$B_i = \{a_i,\, b_i,\, c_{i,1}, \ldots, c_{i,d-1}\}$$

be disjoint sets of $d+1$ vertices carrying all edges of K_{d+1} except a_ib_i. Add two adjacent vertices x and y, join x to every a_i and y to every b_i, and add no other edges. Call the resulting graph $G_{\mathrm{B}}(d)$.

Proposition 4. *$G_{\mathrm{B}}(d)$ is connected and d-regular, with* $|V(G_{\mathrm{B}}(d))| = d^2 + 1$.

Proof. Each a_i has degree $d-1$ inside its block plus the edge to x; each b_i likewise plus the edge to y; each $c_{i,j}$ has its d neighbours inside the block; and each hub is adjacent to $d-1$ block vertices plus the other hub. Every block meets x, and x is adjacent to y, so $G_{\mathrm{B}}(d)$ is connected, with $|V(G_{\mathrm{B}}(d))| = 2 + (d-1)(d+1) = d^2 + 1$. □

Proposition 5. $\gamma_t(G_{\mathrm{B}}(d)) = d + 1$.

Proof. The set $D = \{x, y, a_1, \ldots, a_{d-1}\}$ is total dominating: x and y are adjacent to each other; every a_i is adjacent to x; every b_i is adjacent to y; and every $c_{i,j}$ is adjacent to a_i. Hence $\gamma_t(G_{\mathrm{B}}(d)) \leq d + 1$. Conversely, by Proposition 4 and Lemma 1,

$$\gamma_t(G_{\mathrm{B}}(d)) \;\geq\; \left\lceil \frac{d^2+1}{d} \right\rceil = d + 1. \qquad \square$$

Proposition 6. $p(G_{\mathrm{B}}(d)) = d - 3$.

Proof. Deleting $S = \{x, y\}$ leaves exactly the $d-1$ components $B_1, \ldots, B_{d-1}$, so Lemma 2 gives $p(G_{\mathrm{B}}(d)) \geq (d-1) - 2 = d - 3$. For the upper bound, take the single path

$$Q: \quad b_1,\, c_{1,1}, \ldots, c_{1,d-1},\, a_1,\; x,\; a_2,\, c_{2,1}, \ldots, c_{2,d-1},\, b_2,\; y,\; b_3,\, c_{3,1}, \ldots, c_{3,d-1},\, a_3,$$

which is a path because the edges xa_1, xa_2, yb_2, yb_3 all exist and within each block only $a_i b_i$ is missing. Together with one path $b_i, c_{i,1}, \ldots, c_{i,d-1}, a_i$ for each $i \in \{4, \ldots, d-1\}$, this gives $1 + (d-4) = d-3$ pairwise vertex-disjoint paths covering $V(G_{\mathrm{B}}(d))$. □

Corollary 2. *For $d \geq 8$ the graph $G_{\mathrm{B}}(d)$ is a connected d-regular graph with*

$$\gamma_t(G_{\mathrm{B}}(d)) = d + 1 \; < \; 2d - 6 \; = \; 2p(G_{\mathrm{B}}(d)),$$

so Conjecture 247 fails for every degree $d \geq 8$. At the boundary, $G_{\mathrm{B}}(7)$ is 7-regular on 50 vertices with $\gamma_t = 8 = 2p$ and $p = 4$.

Proof. The inequality $d + 1 < 2(d-3)$ is equivalent to $d > 7$; the boundary value follows from Propositions 4–6 with $d = 7$. □

Theorem 1 follows from Corollaries 1 and 2.

5 Remarks

Remark 1 (Mechanism). Since $\gamma_t(G) \geq n/d$ for every d-regular graph on n vertices (Lemma 1), any counterexample must satisfy $p(G) > n/(2d)$: its path covers must be forced to be inefficient relative to the degree. Both families achieve this in the extremal way, with γ_t equal to the counting bound $\lceil n/d \rceil$ exactly, while a separator of one or two vertices forces p to grow linearly with the number of blocks.

Remark 2 (Equality). West's commentary [7] notes that $\gamma_t = 2p$ holds for disjoint unions of cliques; among connected graphs, every complete graph K_{d+1} attains equality trivially, with $\gamma_t = 2$ and $p = 1$. The equalities in Corollaries 1 and 2 are of a different kind: $G_{\mathrm{A}}(3)$ and $G_{\mathrm{B}}(7)$ attain $\gamma_t = 2p$ with $p = 2$ and $p = 4$ respectively, i.e. far from traceable. Consequently, if the conjecture is true in degree 6 or 7, it is tight there beyond the trivial examples.

Remark 3 (Size of the violation). The additive gap is $2p - \gamma_t = m - 3$ over the one-hub family and $d - 7$ over the two-hub family, so it is unbounded; in both families $\gamma_t/(2p) \to \frac{1}{2}$. We make no claim that $G_{\mathrm{A}}(4)$ is a minimum-order counterexample among all regular graphs; it is only the smallest member of these two families that violates the conjecture.

Question 1. Every graph in the two families satisfies $\gamma_t = p + 2$ or $\gamma_t = p + 4$ respectively, and complete graphs and balanced complete bipartite graphs satisfy $\gamma_t = p + 1$. Is it true that $\gamma_t(G) \geq p(G) + 1$ for every connected regular graph G on at least two vertices? We have not searched the literature for this weaker inequality; it may be known, and the constructions above do not challenge it.

Remark 4 (Independence). The constructions were derived independently. Searches of the maintained listing [3] and of public sources located no prior counterexample to Conjecture 247 at the time of writing; this is not an assertion of absolute priority, since an unpublished or unindexed prior observation may exist.

Remark 5 (Computer verification). A self-contained Python script accompanying this note rebuilds both families from their definitions and checks, for $G_{\mathrm{A}}(m)$ with $2 \leq m \leq 12$ and $G_{\mathrm{B}}(d)$ with $4 \leq d \leq 14$: the order and degree sequence, connectivity, the component counts after deleting the separators, the stated total dominating sets, and the stated path covers (verifying every consecutive pair is an edge and that the paths partition the vertex set). For every member of either family with at most 37 vertices (namely $G_{\mathrm{A}}(m)$ for $m \in \{2, 3, 4\}$ and $G_{\mathrm{B}}(d)$ for $d \in \{4, 5, 6\}$), it also confirms the total domination number exactly, by exhaustive enumeration of all vertex subsets of size less than the claimed value; for $G_{\mathrm{A}}(4)$ this involves all 74,518 subsets of size at most 4. All checks pass. The computation is supplementary: the proofs above are elementary and do not depend on it.

Acknowledgements

The constructions, their verification, and the accompanying formalisation were developed with the assistance of the language models Sol 5.6 and Fable 5. The author is responsible for all claims and for any errors that remain.

A Formal statements in Lean 4

The statements of Conjecture 247, of the Liu–West theorem, of Theorem 1, and of the four remaining open cases have been formalised in Lean 4 over Mathlib, in the conventions of the *formal-conjectures* repository [4], which already lists other conjectures from *Written on the Wall II*. The two invariants are taken from that repository's supporting library (`FormalConjecturesForMathlib`), which defines `SimpleGraph.totalDominationNumber` and `SimpleGraph.pathCoverNumber` with the meanings used in this note, so the file introduces no new definitions beyond a degree-parametrised form `Conjecture247For` of the statement. Proofs are elided with `sorry`: the listing provides machine-readable *statements*, not formal proofs, and the unknown truth values of the open cases are marked `answer(sorry)` in repository style. The hypothesis `Nontrivial` excludes the one-vertex graph, the only connected regular graph without a total dominating set. The listing below omits the repository's licence header and module docstring. The statements type-check against the repository's pinned toolchain (Lean 4 v4.27.0 with the corresponding Mathlib): a `lake build` of the module completes without error, with `sorry` in place of every proof.

```
import FormalConjecturesUtil


namespace WrittenOnTheWallII.GraphConjecture247

open Classical SimpleGraph

/-- The statement of Conjecture 247 restricted to `d`-regular graphs: every
finite simple connected `d`-regular graph on at least two vertices satisfies
`γₜ(G) ≥ 2 · p(G)`.  (`Nontrivial α` excludes the one-vertex graph, for which
the total domination number is undefined and takes the junk value `0`.) -/
def Conjecture247For (d : ℕ) : Prop :=
  ∀ (α : Type) [Fintype α] [DecidableEq α] [Nontrivial α]
    (G : SimpleGraph α) [DecidableRel G.Adj],
    G.Connected → G.IsRegularOfDegree d →
      2 * pathCoverNumber G ≤ totalDominationNumber G
```

```
/--
WOWII [Conjecture 247](http://cms.dt.uh.edu/faculty/delavinae/research/wowII/)
(February 23, 2007), stated as its negation because the conjecture is false:
it is *not* the case that every simple connected regular graph `G` satisfies
`γ_t(G) ≥ 2 · p(G)`.  Disproved by Day (2026); the smallest known
counterexample is `8`-regular on `37` vertices with `γ_t = 5 < 6 = 2 · p`.
-/
@[category research solved, AMS 5]
theorem conjecture247 : ¬ ∀ d : ℕ, Conjecture247For d := by
  sorry

/-- Liu and West (2007): Conjecture 247 holds in every degree `d ≤ 3`; the
cases `d ≤ 2` are elementary and the cubic case is proved in [DLPWW],
Congr. Numer. 185 (2007), 81-95.  West notes it also follows from Reed's
bound `p(G) ≤ (n + 2) / 9` for connected cubic graphs. -/
@[category research solved, AMS 5]
theorem conjecture247.degree_le_three (d : ℕ) (hd : d ≤ 3) :
    Conjecture247For d := by
  sorry

/-- Day (2026): Conjecture 247 fails in every degree `d ≥ 8`.  Witnesses: for
even `d`, one hub joined to `d/2` blocks `K_{d+1} − e` (order
`d^2/2 + d/2 + 1`); for arbitrary `d ≥ 8`, two adjacent hubs joined to `d − 1`
blocks `K_{d+1} − e` (order `d^2 + 1`). -/
@[category research solved, AMS 5]
theorem conjecture247.not_of_eight_le_degree (d : ℕ) (hd : 8 ≤ d) :
    ¬ Conjecture247For d := by
  sorry

/-- The smallest counterexample of Day (2026), as a concrete target for a
formal proof: a connected `8`-regular graph on `37` vertices with total
domination number `5` and path cover number `3`. -/
@[category research solved, AMS 5]
theorem conjecture247.counterexample_on_37_vertices :
    ∃ G : SimpleGraph (Fin 37),
      G.Connected ∧ G.IsRegularOfDegree 8 ∧
        totalDominationNumber G = 5 ∧ pathCoverNumber G = 3 := by
  sorry

/-- Is Conjecture 247 true for `4`-regular graphs?  Open. -/
@[category research open, AMS 5]
theorem conjecture247.degree_four : Conjecture247For 4 ↔ answer(sorry) := by
  sorry

/-- Is Conjecture 247 true for `5`-regular graphs?  Open. -/
@[category research open, AMS 5]
theorem conjecture247.degree_five : Conjecture247For 5 ↔ answer(sorry) := by
  sorry

/-- Is Conjecture 247 true for `6`-regular graphs?  Open.  Equality with path
cover number `2` is attained (`conjecture247.equality_on_22_vertices`), so
the inequality, if true in degree `6`, is tight. -/
@[category research open, AMS 5]
theorem conjecture247.degree_six : Conjecture247For 6 ↔ answer(sorry) := by
  sorry
```

```
/-- Is Conjecture 247 true for ‘7‘-regular graphs?  Open.  Equality with path
cover number ‘4‘ is attained (‘conjecture247.equality_on_50_vertices‘), so
the inequality, if true in degree ‘7‘, is tight. -/
@[category research open, AMS 5]
theorem conjecture247.degree_seven : Conjecture247For 7 ↔ answer(sorry) := by
  sorry

/-- Equality ‘γₜ = 2 · p‘ holds in degree ‘6‘ with path cover number ‘2‘
(Day 2026: three blocks ‘K₇ − e‘ and one hub, ‘22‘ vertices).  Every complete
graph attains equality trivially with ‘p = 1‘; this witness shows tightness
in the non-traceable range where the conjecture has content. -/
@[category research solved, AMS 5]
theorem conjecture247.equality_on_22_vertices :
    ∃ G : SimpleGraph (Fin 22),
      G.Connected ∧ G.IsRegularOfDegree 6 ∧
        totalDominationNumber G = 4 ∧ pathCoverNumber G = 2 := by
  sorry

/-- Equality ‘γₜ = 2 · p‘ holds in degree ‘7‘ with path cover number ‘4‘
(Day 2026: six blocks ‘K₈ − e‘ and two adjacent hubs, ‘50‘ vertices). -/
@[category research solved, AMS 5]
theorem conjecture247.equality_on_50_vertices :
    ∃ G : SimpleGraph (Fin 50),
      G.Connected ∧ G.IsRegularOfDegree 7 ∧
        totalDominationNumber G = 8 ∧ pathCoverNumber G = 4 := by
  sorry

end WrittenOnTheWallII.GraphConjecture247
```

References

[1] E. J. Cockayne, R. M. Dawes, and S. T. Hedetniemi, Total domination in graphs, *Networks* **10** (1980), 211–219.

[2] E. DeLaViña, Q. Liu, R. Pepper, B. Waller, and D. B. West, Some conjectures of Graffiti.pc on total domination, Proceedings of the Thirty-Eighth Southeastern International Conference on Combinatorics, Graph Theory and Computing, *Congressus Numerantium* **185** (2007), 81–95.

[3] E. DeLaViña, *Written on the Wall II: Conjectures of Graffiti.pc*, `https://cms.dt.uh.edu/faculty/delavinae/research/wowII/`. Conjecture 247 is dated February 23, 2007, and is listed as open; accessed 24 July 2026.

[4] Google DeepMind, *formal-conjectures: a collection of formalised statements of mathematical conjectures in Lean*, `https://github.com/google-deepmind/formal-conjectures`.

[5] M. A. Henning and A. Yeo, *Total Domination in Graphs*, Springer Monographs in Mathematics, Springer, New York, 2013.

[6] B. Reed, Paths, stars and the number three, *Combinatorics, Probability and Computing* **5** (1996), 277–295.

[7] D. B. West, *Some Conjectures of Graffiti.pc (2004–07)*, REGS in Combinatorics, University of Illinois, `https://dwest.web.illinois.edu/regs/graffiti.html`. Accessed 24 July 2026.